

# A note on curves equipartition

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The problem of the existence of an equi-partition of a curve in  $\mathbb{R}^n$  has recently been raised in the context of computational geometry (see [2] and [3]). The problem is to show that for a (continuous) curve  $\Gamma : [0, 1] \rightarrow \mathbb{R}^n$  and for any positive integer  $N$ , there exist points  $t_0 = 0 < t_1 < \dots < t_{N-1} < 1 = t_N$ , such that  $d(\Gamma(t_{i-1}), \Gamma(t_i)) = d(\Gamma(t_i), \Gamma(t_{i+1}))$  for all  $i = 1, \dots, N$ , where  $d$  is a metric or even a semi-metric (a weaker notion) on  $\mathbb{R}^n$ .

In fact, this problem, for  $\mathbb{R}^n$  replaced by any metric space  $(X, d)$  was given a positive solution by Urbanik [4] under the (necessary) constraint  $\Gamma(0) \neq \Gamma(1)$ .

We show here that the existence of such points, in a broader context, is a consequence of Brouwer's fixed point theorem (for reference see any intermediate level book on topology, e.g. [1]).

Let  $\Delta^n$  be the  $n$ -dimensional simplex and  $\sigma^n$  be its boundary. Let  $\tau$  be a permutation of  $\{0, \dots, n\}$ . We define a map  $\varphi_\tau : \Delta^n \rightarrow \Delta^n$  in the following way: Every  $x \in \Delta^n$  has a unique representation  $x = \sum_{i=0}^n \alpha_i(x) p_i$ , where  $p_i$  are the vertices of  $\Delta^n$  and  $0 \leq \alpha_i(x) \leq 1$ ,  $\sum_{i=0}^n \alpha_i(x) = 1$ . We then define:

$$\varphi_\tau(x) = \sum_{i=0}^n \alpha_i(x) p_{\tau(i)}.$$

$\varphi_\tau$  is, of course, continuous.

**Lemma 1** *Let  $\tau$  be a cyclic permutation of  $\{0, 1, \dots, n\}$  and let  $\Psi : \sigma^n \rightarrow \sigma^n$  be a map such that, for every proper face  $A$  of  $\Delta^n$ ,  $\Psi(A) \subset \varphi_\tau(A)$ . Then  $\Psi$  has no fixed point.*

**Proof.** Assume that  $\Psi(x) = x$  for some  $x \in \sigma^n$  and let  $x = \sum_{i=0}^n \alpha_i(x) p_i$  be the representation of  $x$ . Let  $A = \text{face}(x)$  be the minimal face of  $\Delta^n$  containing  $x$ . Without loss of generality we may assume that  $A = \text{conv}\{p_0, \dots, p_k\}$  for some  $0 \leq k < n$ . That is,  $x = \sum_{i=0}^k \alpha_i(x) p_i$  and  $\alpha_i(x) > 0$  for  $i = 0, \dots, k$ . Now, by the assumption,  $\Psi(x) \in \varphi_\tau(A)$ , thus  $x = \Psi(x) = \sum_{i=0}^k \beta_i p_{\tau(i)}$ . By the uniqueness of the representation we have:  $\{p_{\tau(0)}, \dots, p_{\tau(k)}\} = \{p_0, \dots, p_k\}$  (and  $\beta_i = \alpha_{\tau^{-1}(i)}$  for all  $0 \leq i \leq k$ ). But, since  $\tau$  is cyclic, no proper subset of  $\{0, \dots, n\}$  is mapped by  $\tau$  on itself. This is a contradiction. ■

Let  $X$  be a non-empty set and  $\Gamma : [0, 1] \rightarrow X$  a function. Let  $d : X \times X \rightarrow \mathbb{R}_+$  (that is  $d(x, y) \geq 0$  for all  $x, y \in X$ ) satisfy  $d(x, x) = 0$ . Assume also that  $d(\Gamma(s), \Gamma(t))$  is continuous on  $[0, 1] \times [0, 1]$  (this last property holds, for example, if  $X$  is a metric space,  $\Gamma$  is continuous and  $d$  is continuous on  $X \times X$ ). Associated with  $d$  as above, we denote, for  $t_1, t_2 \in [0, 1]$ ,  $D(t_1, t_2) = d(\Gamma(t_1), \Gamma(t_2))$ .

Theorem 2 which follows is motivated by the above context, but, as stated, is true for any continuous function  $D : [0, 1] \times [0, 1] \rightarrow \mathbb{R}_+$ . In the particular case that  $D$  is constructed as above, with  $d$  a metric on  $X$ , Theorem 2 implies Urbanik's result [4].

**Theorem 2** *Given a continuous function  $D : [0, 1] \times [0, 1] \rightarrow \mathbb{R}_+$  such that  $D(t, t) = 0$  for all  $t \in [0, 1]$ . Then for every positive integer  $N$  there exist points  $t_0 = 0 \leq t_1 \leq \dots \leq t_{N-1} \leq 1 = t_N$  such that  $D(t_{i-1}, t_i) = D(t_i, t_{i+1})$  for  $i = 1, \dots, N-1$ .*

*Moreover, if for a given  $N$  there are no points  $t_0 = 0 \leq t_1 \leq \dots \leq t_{N-1} \leq 1 = t_N$  such that  $D(t_{i-1}, t_i) = 0$  for  $i = 1, \dots, N-1$ , then for every sequence of positive real numbers  $\alpha_1, \dots, \alpha_N$  such that  $\sum_{i=1}^N \alpha_i = 1$ , there exist points  $t_0 = 0 < t_1 < \dots < t_{N-1} < 1 = t_N$  such that*

$$\frac{D(t_{i-1}, t_i)}{D(t_i, t_{i+1})} = \frac{\alpha_i}{\alpha_{i+1}} \quad \text{for } i = 1, \dots, N-1.$$

**Proof.** If there are points  $t_0 = 0 \leq t_1 \leq \dots \leq t_{N-1} \leq 1 = t_N$  such that  $D(t_{i-1}, t_i) = 0$  for  $i = 1, \dots, N-1$ , then the first claim is obvious. Thus let us assume that such a sequence does not exist and prove the ‘‘Moreover’’ assertion.

Let  $S$  be the  $(N-1)$ -dimensional simplex

$$S = \{(t_1, \dots, t_{N-1}) \mid 0 \leq t_1 \leq \dots \leq t_{N-1} \leq 1\}$$

and let  $F : S \rightarrow \mathbb{R}^N$  be defined by

$$F(t_1, \dots, t_{N-1}) = (D(0, t_1), D(t_1, t_2), \dots, D(t_{N-1}, 1)).$$

Then,  $F$  is continuous and  $F(S) \subset \mathbb{R}_+^N$ . Moreover, every proper face  $A$  of  $S$  is characterized as being the set of  $(N-1)$ -tuples  $(t_1, \dots, t_{N-1}) \in S$  such that, in the chain of inequalities  $0 \leq t_1 \leq \dots \leq t_{N-1} \leq 1$ , there are  $0 < k \leq N-1$  equalities (the dimension of  $A$  is then  $N-1-k$ ). Thus, for  $0 < k \leq N-1$  and a  $(N-1-k)$ -dimensional face  $A$  of  $S$ ,  $F(A)$  is contained in the subspace  $\mathcal{A}$  of  $\mathbb{R}^n$ :

$$\mathcal{A} = \{(x_1, \dots, x_N) \mid x_i = 0 \text{ whenever } t_{i-1} = t_i \text{ for all } (t_1, \dots, t_{N-1}) \in A\}$$

(here, and in the sequel,  $t_0 = 0$  and  $t_N = 1$ ).

Let  $\Sigma$  be the  $(N-1)$ -dimensional simplex which is the convex hull in  $\mathbb{R}^N$  of the unit coordinate vectors  $e_i$ ,  $i = 1, \dots, N$ . Let  $G : \mathbb{R}_+^N \setminus \{\bar{0}\} \rightarrow \Sigma$  be the radial projection with center  $\bar{0} \in \mathbb{R}^N$ . The composed map  $GF$  maps  $S$  continuously into  $\Sigma$  (as we have excluded the possibility that  $\bar{0} \in F(S)$ ).

**Claim** *The relative interior of  $\Sigma$  is contained in  $GF(S)$ .*

We remark that, by simple induction, it follows from the above Claim that, in fact,  $GF$  maps  $S$  onto  $\Sigma$ , but the Claim as is, is clearly sufficient to complete the proof of Theorem 2,

since it implies that, for any positive  $\alpha_1, \dots, \alpha_N$  with  $\sum_{i=1}^N \alpha_i = 1$ , the ray  $L$  from the origin in  $\mathbb{R}^N$  with parametric equation

$$L : \bar{x} = (\alpha_1 t, \dots, \alpha_N t), \quad t > 0,$$

that intersects  $\Sigma$  at the point  $(\alpha_1, \dots, \alpha_N)$ , must, by the Claim, intersect also  $F(S)$ .

**Proof of the Claim.** Assume, for contradiction, that there exists a point  $P$  in the relative interior of  $\Sigma$ , such that  $P \notin GF(S)$ . Let

$$H : \Sigma \setminus \{P\} \rightarrow \partial\Sigma$$

be the radial projection onto  $\partial\Sigma$  with center  $P$  (in the hyperplane containing  $\Sigma$ ).  $H$  is continuous on  $\Sigma \setminus \{P\}$  and, since  $P \notin GF(S)$ , the map  $HGF : S \rightarrow \partial\Sigma$  is continuous.  $H$  restricted to  $\partial\Sigma$  is the identity, thus, by the previous discussion,  $HGF(A) \subset \Sigma \cap \mathcal{A}$  for every proper face  $A$  of  $S$ . Let us identify now  $S$  with  $\Sigma$  by the natural affine homeomorphism that identifies every face  $A$  of  $S$  with  $\Sigma \cap \mathcal{A}$ . Via this identification,  $HGF$  induces a continuous map

$$\Phi : \Sigma \rightarrow \partial\Sigma$$

with the property that  $\Phi(A) \subset A$  for every proper face  $A$  of  $\Sigma$ .

Now let  $\varphi_\tau : \Sigma \rightarrow \Sigma$  be the map associated with a cyclic permutation of the vertices of  $\Sigma$ , as in the context of Lemma 1, and let

$$\Psi = \varphi_\tau \Phi : \Sigma \rightarrow \partial\Sigma.$$

Then  $\Psi$  restricted to  $\partial\Sigma$  satisfies the assumptions of Lemma 1, hence  $\Psi$  has no fixed point in  $\partial\Sigma$ . Considering  $\Psi$  as a map from  $\Sigma$  into  $\Sigma$  we get a continuous map from  $\Sigma$  into itself that has no fixed point. This contradicts Brouwer's fixed point theorem. ■

**Reminder:** Brouwer's fixed point theorem states that any continuous map from  $B^K$  into  $B^K$  has a fixed point ( $B^K$  is the closed unit ball of  $\mathbb{R}^K$  but, of course, can be replaced by anything homeomorphic to it, like  $\Sigma$  here).

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## References

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